C3 JUNE 2013 INT

1. Express

$$\frac{3x+5}{x^2+x-12} - \frac{2}{x-3}$$

as a single fraction in its simplest form.

$$\frac{3x+5}{(x+4)(x-3)} - \frac{2(x+4)}{(x+4)(x-3)} = \frac{3x+5-2x-8}{(x+4)(x-3)}$$

$$= \frac{x-3}{(x+4)(x-3)} = \frac{1}{x+4}$$

Q1) MUCH EASIER THAN UK PAPER

2.

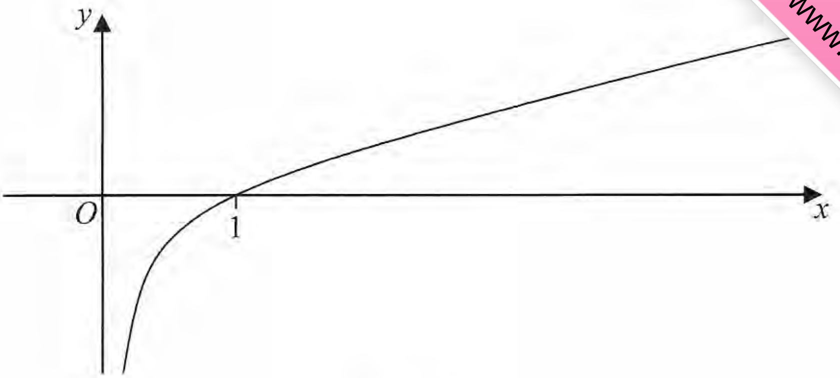


Figure 1

Figure 1 shows a sketch of the curve with equation $y = \mathrm{f}(x)$, $x > 0$, where f is an increasing function of x. The curve crosses the x-axis at the point $(1, 0)$ and the line $x = 0$ is an asymptote to the curve.

On separate diagrams, sketch the curve with equation

(a) $y = \mathrm{f}(2x)$, $x > 0$

(2)

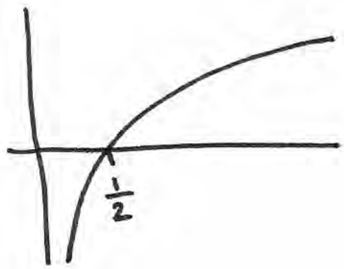
(b) $y = |\mathrm{f}(x)|$, $x > 0$

(3)

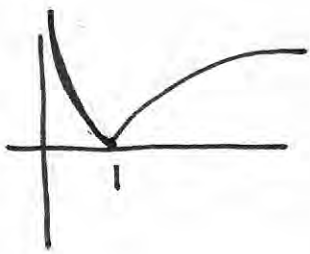
Indicate clearly on each sketch the coordinates of the point at which the curve crosses or meets the x-axis.

a) $y = f(2x)$

$\rightarrow 2 \leftarrow$



b) $y = |f(x)|$



Q2. Slightly easier than uk. No asymptotes

3. $$f(x) = 7\cos x + \sin x$$

Given that $f(x) = R\cos(x - \alpha)$, where $R > 0$ and $0 < \alpha < 90°$,

(a) find the exact value of R and the value of α to one decimal place.

(b) Hence solve the equation

$$7\cos x + \sin x = 5$$

for $0 \leqslant x < 360°$, giving your answers to one decimal place.

(5)

(c) State the values of k for which the equation

$$7\cos x + \sin x = k$$

has only one solution in the interval $0 \leqslant x < 360°$

(2)

a) $R\cos(x-\alpha) = R\cancel{\cos x}\cos\alpha + R\cancel{\sin x}\sin\alpha$

$7\cancel{\cos x} \quad + \quad 1\cancel{\sin x}$

$\Rightarrow \dfrac{R\sin\alpha = 1}{R\cos\alpha = 7} \Rightarrow \tan\alpha = \frac{1}{7} \Rightarrow 8.1°$ $\quad R^2 = 1^2 + 7^2$

$\Rightarrow R^2 = 50$

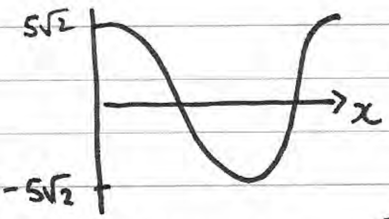
$\Rightarrow R = 5\sqrt{2}$

b) $5\sqrt{2}\cos(x - 8.1) = 5$

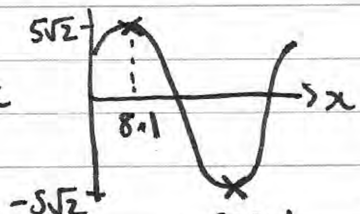
$\Rightarrow \cos(x - 8.1) = \dfrac{5}{5\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ $\quad$ ~~$\Rightarrow x \cos(x-8.1) =$~~

$\Rightarrow x - 8.1 = 45, 315 \quad \therefore \quad x = 53.1;\ 323.1°$

c) $5\sqrt{2}\cos(x - 8.1) = k$



$5\sqrt{2}\cos x$



278.1

$\therefore\ k = 5\sqrt{2}, -5\sqrt{2}$

$x = 8.1, 278.1$
(no asked)

Q3. MUCH! easier than uk.

4. The functions f and g are defined by

$$f : x \mapsto 2|x| + 3, \qquad x \in \mathbb{R},$$

$$g : x \mapsto 3 - 4x, \qquad x \in \mathbb{R}$$

(a) State the range of f.
(2)

(b) Find fg(1).
(2)

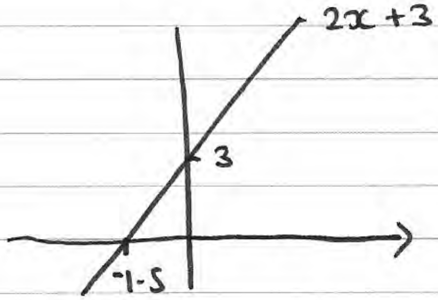
(c) Find g^{-1}, the inverse function of g.
(2)

(d) Solve the equation

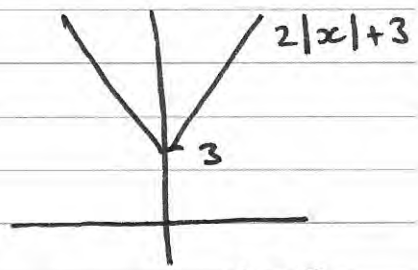
$$gg(x) + [g(x)]^2 = 0$$

(5)

a)



$\Rightarrow$



$\therefore$ range $y \geq 3$
$f(x) \geq 3$

b) $g(1) = -1 \quad \therefore \quad fg(1) = f(-1) = 5$

c) $y = 3 - 4x \Rightarrow x = 3 - 4y \Rightarrow 4y = 3 - x$

$\therefore y = g^{-1} = \dfrac{3-x}{4}$

d) $gg(x) = 3 - 4(3 - 4x) = 3 - 12 + 16x = 16x - 9$

$[g(x)]^2 = (3-4x)^2 = 9 - 24x + 16x^2$

$\Rightarrow 16x - \cancel{9} + \cancel{9} - 24x + 16x^2 = 0$

$\Rightarrow 16x^2 - 8x = 0$

$\Rightarrow 8x(2x - 1) = 0$

$\therefore x = 0$
$x = \frac{1}{2}$

[Q4 Easier than uk.]

5. (a) Differentiate

$$\frac{\cos 2x}{\sqrt{x}}$$

with respect to x.

(3)

(b) Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\sec^2 3x)$ can be written in the form

$$\mu(\tan 3x + \tan^3 3x)$$

where μ is a constant.

(3)

(c) Given $x = 2\sin\left(\frac{y}{3}\right)$, find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, simplifying your answer.

(4)

a) $u = \cos 2x \qquad v = x^{\frac{1}{2}}$

$u' = -2\sin 2x \qquad v' = \frac{1}{2}x^{-\frac{1}{2}}$

$\Rightarrow \frac{dy}{dx} = \dfrac{-2\sqrt{x}\sin 2x - \dfrac{\cos 2x}{2\sqrt{x}}}{x}$

b) $y = (\sec 3x)^2 \Rightarrow \frac{dy}{dx} = 2(\sec 3x) \times 3\sec 3x \tan 3x$

$= 2\sec^2 3x \tan 3x$

$\Rightarrow (2\tan^2 3x + 2)\tan 3x \quad = 2(\tan^3 3x + \tan 3x)$

\#

c) $x = 2\sin(\frac{1}{3}y)$

$\Rightarrow \frac{dx}{dy} = \frac{2}{3}\cos(\frac{1}{3}y) \Rightarrow \frac{dy}{dx} = \frac{3}{2\cos(\frac{1}{3}y)}$

$= \frac{3}{2\sqrt{\cos^2(\frac{1}{3}y)}} = \frac{3}{2\sqrt{1-\sin^2(\frac{1}{3}y)}} \quad \therefore \frac{dy}{dx} = \frac{3}{2\sqrt{1-(\frac{x}{2})^2}}$

alt. $\frac{dy}{dx} = \frac{3}{2}\sec(\frac{y}{3}) = \frac{3}{2}\sec\left[\arcsin(\frac{x}{2})\right]$

(Q5. c) isn't easy, but overall, easier and more standard than UK.)

6. (i) Use an appropriate double angle formula to show that

$$\operatorname{cosec} 2x = \lambda \operatorname{cosec} x \sec x,$$

and state the value of the constant λ.

(ii) Solve, for $0 \leqslant \theta < 2\pi$, the equation

$$3\sec^2\theta + 3\sec\theta = 2\tan^2\theta$$

You must show all your working. Give your answers in terms of π.

(6)

i) $\dfrac{1}{\sin 2x} = \dfrac{1}{2\sin x\cos x} = \dfrac{1}{2}\operatorname{cosec} x \sec x$

$\lambda = \frac{1}{2}$

ii) $3\sec^2\theta + 3\sec\theta = 2(\sec^2 x - 1)$

$\Rightarrow 3\sec^2\theta + 3\sec\theta = 2\sec^2 x - 2$

$\Rightarrow \sec^2\theta + 3\sec\theta + 2 = 0$

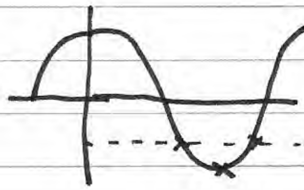
$\Rightarrow (\sec\theta + 2)(\sec\theta + 1) = 0$

$\sec\theta = -2 \qquad \sec\theta = -1$

$\cos\theta = -\frac{1}{2} \qquad \cos\theta = -1$

$\therefore \theta = 120°, 240°, 180°$

$= \frac{2\pi}{3}, \frac{4\pi}{3}, \pi$



Q6. V. easy, standard, trig qu.

overall trig was MUCH easier on the paper

7.

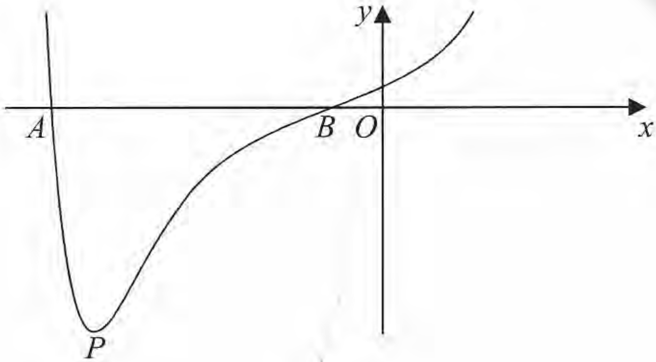


Figure 2

Figure 2 shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$ where

$$\mathrm{f}(x) = (x^2 + 3x + 1)\mathrm{e}^{x^2}$$

The curve cuts the x-axis at points A and B as shown in Figure 2.

(a) Calculate the x coordinate of A and the x coordinate of B, giving your answers to 3 decimal places.

(2)

(b) Find $\mathrm{f}'(x)$.

(3)

The curve has a minimum turning point at the point P as shown in Figure 2.

(c) Show that the x coordinate of P is the solution of

$$x = -\frac{3(2x^2 + 1)}{2(x^2 + 2)}$$

(3)

(d) Use the iteration formula

$$x_{n+1} = -\frac{3(2x_n^2 + 1)}{2(x_n^2 + 2)}, \qquad \text{with } x_0 = -2.4,$$

to calculate the values of x_1, x_2 and x_3, giving your answers to 3 decimal places.

(3)

The x coordinate of P is α.

(e) By choosing a suitable interval, prove that $\alpha = -2.43$ to 2 decimal places.

(2)

a) $x^2+3x+1=0$ $\qquad \left(x+\frac{3}{2}\right)^2-\frac{9}{4}+\frac{}{4}$

$\Rightarrow \left(x+\frac{3}{2}\right)^2=\frac{5}{4} \qquad \Rightarrow x+\frac{3}{2}=\pm\frac{\sqrt{5}}{2} \qquad \therefore\ x=$

$x_A=\frac{-3-\sqrt{5}}{2}=\underline{-2.618} \qquad x_B=\frac{-3+\sqrt{5}}{2}=\underline{-0.38}$

b) $u=x^2+3x+1 \qquad v=e^{x^2}$

$u'=2x+3 \qquad v'=2xe^{x^2}$

$\therefore\ f'(x)=(2x+3)e^{x^2}+2x(x^2+3x+1)e^{x^2}$

c) TP when $f'(x)=0$

$(2x+3+2x^3+6x^2+2x)e^{x^2}=0$

$\Rightarrow\ 2x^3+6x^2+4x+3=0$

$\Rightarrow\ 2x^3+4x=-6x^2-3$

$\Rightarrow\ x\times 2(x^2+2)=-3(2x^2+1)$

$\therefore\ x=-\frac{3(2x^2+1)}{2(x^2+2)}$ #

d) $x_0=-2.4$

$x_1=-2.420$

$x_2=-2.427$

$x_3=-2.430$

e) $f'(-2.425)=22.\ >0$ $\qquad \therefore$ by sign change

$f'(-2.435)=-15\ <0$ $\qquad$ rule $\alpha=-2.43$ (2dp)

Q7 - similar difficulty

8.

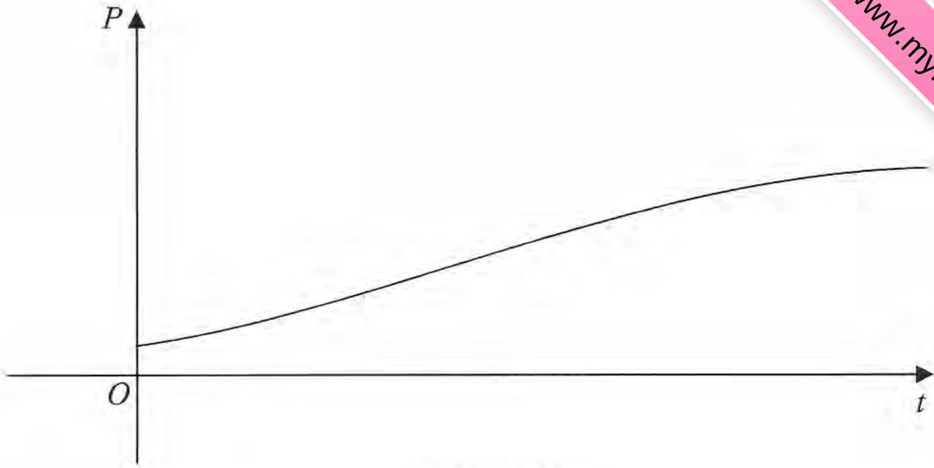


Figure 3

The population of a town is being studied. The population P, at time t years from the start of the study, is assumed to be

$$P = \frac{8000}{1 + 7e^{-kt}}, \qquad t \geqslant 0,$$

where k is a positive constant.

The graph of P against t is shown in Figure 3.

Use the given equation to

(a) find the population at the start of the study,

(2)

(b) find a value for the expected upper limit of the population.

(1)

Given also that the population reaches 2500 at 3 years from the start of the study,

(c) calculate the value of k to 3 decimal places.

(5)

Using this value for k,

(d) find the population at 10 years from the start of the study, giving your answer to 3 significant figures.

(2)

(e) Find, using $\frac{dP}{dt}$, the rate at which the population is growing at 10 years from the start of the study.

(3)

a) $t=0 \Rightarrow P = \frac{8000}{8} = \underline{1000}$

b) as $t \to \infty$ $\quad 7e^{-kt} \to 0 \quad \therefore P \to \frac{8000}{1} = \underline{8000}$

c) $25\cancel{00} = \frac{80\cancel{00}}{1+7e^{-3k}} \Rightarrow 25 + 175e^{-3k} = 80$

$\Rightarrow 175e^{-3k} = 55 \Rightarrow e^{-3k} = \frac{11}{35} \Rightarrow -3k = \ln\left(\frac{11}{35}\right)$

$\therefore k = -\frac{1}{3}\ln\left(\frac{11}{35}\right) = \underline{0.386}$

d) $\frac{8000}{1+7e^{-0.386 \times 10}} = \underline{6970}$

e) $P = 8000(1+7e^{-kt})^{-1}$

$\Rightarrow \frac{dP}{dt} = -8000(1+7e^{-kt})^{-2} \times (-7ke^{-kt})$

$\frac{dP}{dt} = \frac{56000ke^{-kt}}{(1+7e^{-kt})^2}$

at $t=10$ $\quad \frac{dP}{dt} = 346.2$

[e) is awkward but otherwise a reasonable question]

This paper was <u>MUCH</u> easier than uk !